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NOTE

THE SIZE OF 3-CROSS-FREE FAMILIES*

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We give a short and simple proof for the theorem that the size of a 3-cross-free family is linear in the size of the groundset. A family is 3-cross-free if it has no 3 pairwise crossing members.

1. Introduction

We shall prove that if $\mathcal{F}\subset 2^V$ is a 3-cross-free family then $|\mathcal{F}|\le 10|V|$. We call $\mathcal{F}\subset 2^V$ to be *k-cross-free* if $\mathcal{F}$ has no k pairwise crossing members. Sets $A,B\subset V$ are *crossing* if none of $A\cap B, A\setminus B, B\setminus A$ and $V\setminus(A\cup B)$ is empty.

It was conjectured by Karzanov and Lomonosov that $|\mathcal{F}|=O(kn)$ if $\mathcal{F}$ is k-cross-free and $|V|=n$. For $k=2$, this is trivial from the well-known tree representation of laminar families. In [7], Pevzner gave a quite complicated and lengthy proof for the case $k=3$. In Section 2, we present a direct and easy proof for this result. Actually, we prove a slightly more general theorem than the one indicated above. We call a family $\mathcal{F}\subset 2^V$ *weakly k-cross-free with respect to* $a\in V$, if for every $b\in V\setminus\{a\}$ there are no k pairwise crossing members of $\mathcal{F}$ separating a from b. We say that X *separates a from b* if it contains exactly one of them. We call a family *weakly k-cross-free* if it is weakly k-cross-free with respect to some element a of V. In Section 2, we will show that the size of a weakly 3-cross-free family is at most $10n$.

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As far as we know, the conjecture of Karzanov and Lomonosov is still open for $k>3$ and the best known bound is $|\mathcal{F}|=O(kn\log n)$ due to Lomonosov. However, recently there has been some new results obtained by Dress *et al.* [2,1], where all maximum-size 3-cross-free families are described and the conjecture is proved for so-called cyclic 4-cross-free families. There, *cyclic* means that there is a cyclic order on V such that any element of $\mathcal{F}$ is an interval in it. They also show that contrary to some naive expectations, maximum-size k-cross-free families are not cyclic for $k=4$.

The background of the investigation of 3-cross-free families is the so called locking theorem of Karzanov and Lomonosov [4,5] (see also [3]): a family $\mathcal{F}\subset 2^V$ is lockable if and only if $\mathcal{F}$ is 3-cross-free. Family $\mathcal{F}$ is called *lockable* if whenever $G=(W,E)$ is an undirected graph with $V\subset W$ then there exists a fractional path-packing (i.e. a multiflow) f in G such that every $X\in\mathcal{F}$ is locked in G by f. Subset X of V is *locked in G by f* if the total f-value of paths between X and $V\setminus X$ equals the minimum size of the edge-cuts of G separating X from $V\setminus X$. The following stronger version was also proved in [4,5] (for a shorter proof see [6]): $\mathcal{F}$ is 3-cross-free if and only if for any $G=(W,E)$ inner Eulerian graph (that is, $V\subseteq W$ and the degrees of all vertices of $W\setminus V$ are even) there is a collection $\mathcal{P}$ of edge-disjoint paths of G in such a way that for any $X\in\mathcal{F}$, $\mathcal{P}$ contains maximum number of paths connecting X to $V\setminus X$.

2. Weakly 3-cross-free families

In this section we prove our result. Throughout we use the following notation:

$$\mathcal{F}/v := \{X\setminus\{v\} : X\in\mathcal{F}\}$$
$$\mathcal{F}(v) := \{X : v\in X\in\mathcal{F}\ni X\setminus\{v\}\}$$

Theorem 1. *Let $|V|=n\in\mathbb{N}$ and let $\mathcal{F}\subset 2^V$ be a weakly 3-cross-free family. Then $|\mathcal{F}|\le 10n$.*

Proof. Assume to the contrary that $\mathcal{F}$ is a counterexample with $|V|$ minimal, that is, $|\mathcal{F}|>10n$ and $\mathcal{F}$ is weakly 3-cross-free with respect to a. Let us define $\mathcal{F}':=\{X\in\mathcal{F}:a\notin X\}\cup\{V\setminus X:a\in X\in\mathcal{F}\}$. Clearly, $|\mathcal{F}'|>5n$ with the property that

(1) *if $X,Y,Z\in\mathcal{F}'$ with $X\cap Y\cap Z\ne\emptyset$ then X,Y,Z cannot pairwise cross.*

Next we prove:

(2) *For each $x\in V\setminus a$, there exist $A_x,B_x\in\mathcal{F}'(x)$ such that $B_x\ne A_x\subset B_x$ and* $|A_x|\ge 3$.

If $\{x\}\ne P\subset Q\subset R$ is a chain of three different elements from $\mathcal{F}'(x)$, then $A_x=Q$, $B_x=R$ suffices. Otherwise each element of $\mathcal{F}'(x)\setminus\{x\}$ is either inclusionwise minimal or maximal. By (1), we see that $\mathcal{F}'(x)\setminus\{x\}$ contains at most two maxima and at most two minima, hence altogether $|\mathcal{F}'(x)| \le 5$. As $|\mathcal{F}(x)| \le 2|\mathcal{F}'(x)|$, we get that $|\mathcal{F}/x| = |\mathcal{F}| - |\mathcal{F}(x)| > 10|V| - 2|\mathcal{F}'(x)| \ge 10|V \setminus \{x\}|$. This contradicts to the minimality assumption as $\mathcal{F}/x$ is also a weakly 3-cross-free family with respect to a. This proves (2).

Choose $x \in V \setminus a$, such that $|B_x|$ is as small as possible. Let $y, z \in A_x \setminus \{x\}$ be different elements. Observe that $y \in A_x \cap (B_x \setminus \{x\}) \cap B_y$ and that A_x crosses $B_x \setminus \{x\}$. By the choice of x, $|B_y| \ge |B_x|$, hence B_y must contain A_x or $B_x \setminus \{x\}$ by (1). In particular, $z \in A_x \setminus \{x\} \subset B_y$ holds. Then $z \in A_x \cap (B_x \setminus \{x\}) \cap (B_y \setminus \{y\})$, and these three sets of $\mathcal{F}'$ pairwise cross, contradicting (1). ∎

3. Conclusions

As indicated, Karzanov's conjecture about the linear size of k-cross-free families is still open for $k > 3$. However Lomonosov's argument is also valid in our weakly k-cross-free setting. Indeed, let $\mathcal{F}^i := \{X \in \mathcal{F}' : |X| = i\}$ for $i = 0, 1, \ldots, n$, where $\mathcal{F}'$ is defined as in the proof of Theorem 1. Clearly, for every $v \in V \setminus a$ there are less than k sets in $\mathcal{F}^i$ covering v, hence $|\mathcal{F}| \le 2|\mathcal{F}'| = 2\sum_{i=0}^{n} |\mathcal{F}^i| < 2\left(1 + \sum_{i=1}^{n} \frac{kn}{i}\right) = O(kn \log n)$.

Pevzner published a paper about the linear size of 3-cross-free families [7], in which he explores important properties of k-cross-free and 3-cross-free families. Although the proof is not easy to read, he had some interesting remarks that are worth citing. In our terminology his question is the following:

> Is it true that any k-cross-free family on n elements can be decomposed into r $(k-1)$-cross-free families (r is independent of $n, k > 3$)?

He also observes:

> It is possible to show that for $k = 3$ the answer to the above problem is negative (an example of an r-indecomposable 3-cross-free family is a family of stars in a graph without triangles with a chromatic number exceeding r).

It is interesting to see that the answer to Pevzner's question is negative even for all k if we ask it for families that are weakly k-cross-free with respect to a fixed point of the groundset: Let $[n] := \{i \in \mathbb{N} : 1 \le i \le n\}$; $\binom{[n]}{k} := \{X \subset [n] : |X| = k\}$ and define $\mathcal{F}([n], k) := \{\{X \in \binom{[n]}{k} : i \in X\} : i \in [n]\} \subset 2^{\binom{[n]}{k}}$. Although for $k \ge 2$, $n \ge 4$ and $X \in \binom{[n]}{k}$ family $\mathcal{F}([n], k)_X := \{F \in \mathcal{F}([n], k) : X \notin F\}$ consists of pairwise crossing sets, it is already weakly $(k+1)$-cross-free with respect to X. Moreover, any k elements of $\mathcal{F}([n], k)_X$ separate X from another element Y of $\binom{[n]}{k}$, hence for $n \ge (c+1) \cdot k$ it is not possible to partition $\mathcal{F}([n], k)_X$ into c families that are all weakly k-cross-free with respect to X.

Our last remark is that Theorem 1 is not very far from the best possible bound: notice that $\mathcal{F}[n,k] := \{i+[j],[i]+j,[n]\setminus(i+[j]),[n]\setminus([i]+j) : i+1\in[k], j\in[n-i]\}\subset 2^{[n]}$ (where $a+[b]:=[a+b]\setminus[a]$) is a k-cross-free family with roughly $4(k-1)n$ members. In particular, there is a 3-cross-free family $\mathcal{F}[n,3]$ with roughly $8n$ members.

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